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THE MOTION OF A MATERIAL POINT IN FRIEDMANN-LOBACHEVSKY SPACE*

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Some simple instances of the motion of a material point in Friedmann-Lobachevsky space /1/ are constructed and investigated, on the assumption that the space, like Galilean space, is empty (i.e., contains no matter) and that the forces acting on the point, including the gravitational forces, constitute a factor extraneous to the space. Thus, the problem is being considered in the context of a rather unusual "relativistic" mechanics, distinct from relativisitc mechanics proper. As will be seen later, the difference is quantitatively small and can be regulated by slowly varying cosmological factors in the pseudo-Euclidean metric of the space of special relativity theory.

1. Statement of the problem. We consider a Riemannian space with the metric /1/

$$ds^2 = (1 - \alpha/\tau)^4(c^2dt^2 - dx^2 - dy^2 - dz^2) \tag{1.1}$$
$$\tau^2 = t^2 - r^2/c^2,\ r^2 = x^2 + y^2 + z^2, \qquad \tau > 0$$

where c is the speed of light and α is a cosmological constant.

V.A. Fock called this space Friedmann-Lobachevsky space.

Astrophysical observations enable one to calculate the constant and the value of the variable quantity τ for the current epoch /1/,

$$\alpha = 6\times10^8 \text{ years}, \tau^* = 6\times10^9 \text{ years} \tag{1.2}$$

The problem considered here is to construct dynamical equations for a material point; it is also required to find the most interesting motions of the point:

a) inertial motions (geodesics of the Riemannian space with metric (1.1);

b) motions in the field of a central Newtonian force (this is most rationally done by describing the evolution of circular orbits).

The variables x, y, z in the metric (1.1) are assumed to be the Cartesian coordinates of the point, and t the time as measured on the clock of a "stationary" observer.

2. Equations of motion. According to a well-known variational principle, the real motions of a material point give the action functional an extremum value. This may mean, in particular, that the symmetry group of the equations of motion preserves the Lagrange function. On the other hand, by the principle of relativity the equations of motion must be independent of the choice of the inertial reference system. Postulating that the transition between inertial reference systems is effected by transformations which preserve the metric of the Riemannian space (i.e., by the group of motions of this space), we obtain the well-known connection between the interval ds and the Lagrange function. With this done, we then obtain the equations of motion as the usual Lagrange equations with non-potential forces on the right. This very approach is implemented in relativistic dynamics /2/,

$$L = \lambda\sqrt{ds^2/dt^2} = \lambda\,(1 - \alpha/\tau)^2\sqrt{c^2 - V^2},\ V^2 = x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}$$

The constant factor λ must be so chosen that in the limit of classical mechanics, when $c \to \infty$ $\tau^* \to \infty$, the Lagrangian becomes (apart from the factor mc^2) the ordinary kinetic energy of the point: $L = mV^2/2$. We thus obtain

$$L = -mc\left(\frac{1-\alpha/\tau}{1-\alpha/\tau^*}\right)^2\sqrt{c^2 - V^2} \tag{2.1}$$

The equations of motion are now obtained as the usual Lagrange equations. We write them as a single vector equation:

$$\frac{d}{dt}\left[\left(1-\frac{\alpha}{\tau}\right)^2\left(1-\left(\frac{V}{c}\right)^2\right)^{-1/2} m\mathbf{V}\right] = \frac{2m\alpha}{\tau^3}\left(1-\frac{\alpha}{\tau}\right)\sqrt{1-\left(\frac{V}{c}\right)^2}\,\mathbf{r} + \left(1-\frac{\alpha}{\tau^*}\right)^2\mathbf{F} \tag{2.2}$$
$$(1 - \alpha/\tau^*)^2 = 0.81$$

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Here m is the mass of the point, r is its radius-vector, V is its velocity and F is the applied force.

Eq.(2.2) readily implies the moment theorem relative to the origin:

$$\frac{d}{dt}\left[\left(1-\frac{\alpha}{\tau}\right)^2\left(1-\left(\frac{V}{c}\right)^2\right)^{-1/2}(\mathbf{r}\times m\mathbf{V})\right] = 0.81\mathbf{r}\times\mathbf{F} \tag{2.3}$$

The moment theorem is not valid for other points of the space. This selectivity of the origin is due to the fact that not all points of the geometrical space are equivalent in this mechanical model. The mathematical expression of this fact is that the group of motions of the space (1.1) does not contain the translations as a subgroup.

It follows from formula (2.3) that when there are no impressed forces $(\mathbf{F}=0)$, or when the impressed forces reduce to a central force directed toward the origin $(\mathbf{r}\times\mathbf{F}=0)$, one has a first integral - an analogue of the area integral:

$$(1-\alpha/\tau)^2(1-(V/c)^2)^{-1/2}(\mathbf{r}\times m\mathbf{V}) = \mathbf{l} = \text{const} \tag{2.4}$$

This relation clearly shows that the motion of the material point makes place in a plane.

3. ***Inertial motions of a material point.*** The equations of free motion of the point (the equations of the geodesics) are obtained from (2.2) by setting $\mathbf{F}=0$ and developing the left-hand side of the equation by differentiation:

$$(1-\alpha/\tau)\,d\mathbf{V}/dt = 2\alpha\tau^{-3}(1-(V/c)^2)(\mathbf{r}-\mathbf{V}t) \tag{3.1}$$

Rewriting this equation in the form

$$d(\mathbf{r}-\mathbf{V}t)^2/dt = -4\alpha t\tau^{-3}(1-(V/c)^2)(\mathbf{r}-\mathbf{V}t)^2$$

we readily deduce some properties of the geodesics:

a) Trajectories passing through the origin are straight lines, and motion along them is uniform: $\mathbf{r}=\mathbf{V}t$, $\mathbf{V}=\text{const}$.

b) Every trajectory has a certain asymptote

$$\mathbf{r}-\mathbf{V}_\infty t = \mathbf{a},\quad \mathbf{V}_\infty = \lim_{t\to\infty}\mathbf{V},\quad \mathbf{a}=\text{const}.$$

It has proved possible to integrate Eq.(3.1) explicitly, in parametric form (where τ is the parameter):

$$\mathbf{r} = \tau\tau_0^{-1}(\mathbf{r}_0\operatorname{ch}\psi(\tau)+\boldsymbol{\lambda}_0\operatorname{sh}\psi(\tau)),\quad t=\tau\tau_0^{-1}(p_0\operatorname{sh}\psi(\tau)+t_0\operatorname{ch}\psi(\tau)) \tag{3.2}$$

$$\psi(\tau) = \int_{\tau_0}^{\tau}\frac{d\xi}{\sqrt{k^2(\xi-\alpha)^4+\xi^2}},\quad \tau_0\leqslant\tau<\infty$$

$$\boldsymbol{\lambda}_0 = (c^2\tau_0^2\mathbf{v}+(\mathbf{r}\mathbf{v}-c^2t)\mathbf{r}_0)Q^{-1/2},\quad p_0 = (t\mathbf{r}_0\mathbf{v}-r_0^2)Q^{-1/2}$$

$$k^2 = c^2(c^2-v^2)(1-\alpha/\tau_0)^4Q^{-1},\quad Q = (\mathbf{r}_0\mathbf{v}-c^2t_0)^2-c^2\tau_0^2(c^2-v^2)$$

$$\mathbf{r}_0 = \mathbf{r}|_{\tau=\tau_0},\quad \mathbf{v}=\mathbf{V}|_{\tau=\tau_0},\quad t_0 = t|_{\tau=\tau_0}$$

That (3.2) is indeed a solution may be verified by direct substitution.

The process by which (3.1) was integrated consisted of two steps: integration of the one-dimensional equation of motion,

$$x^{\cdot\cdot} = 2\alpha\tau^{-3}(1-\alpha/\tau)^{-1}(1-(x^{\cdot}/c)^2)(x-tx^{\cdot}) \equiv f \tag{3.3}$$

and construction of the full-dimensional solution from this result.

Both steps made essential use of the symmetry group of Eq.(3.1). Examination of the metric (1.1) shows that the group of motions contains the subgroup of four-dimensional complex rotations of the Lorentz group, i.e., it differs from the latter in that it does not contain the translations of space-time. The equation

$$X\varphi = x\frac{\partial\varphi}{\partial t}+c^2t\frac{\partial\varphi}{\partial x}+(c^2-x^{\cdot 2})\frac{\partial\varphi}{\partial x^{\cdot}} = 0$$

constructed using the operator X of the one-dimensional Lorentz group, extended to the derivative $x^{\cdot}$, has first integrals

$$\omega_1 = \tau^2 = t^2-(x/c)^2,\quad \omega_2 = (t+x/c)^2(c-x^{\cdot})/(c+x^{\cdot})$$

The solution of Eq.(3.1) necessarily depends only on the functions ω_1, ω_2.

Considering the equation

$$\frac{\partial\varphi}{\partial t}+x^{\cdot}\frac{\partial\varphi}{\partial x}+f\frac{\partial\varphi}{\partial x^{\cdot}} = 0$$

we introduced new variables $\Omega_1 = \alpha\tau^{-1}$, $\Omega_2 = \omega_2\tau^{-2}$, to obtain a first integral of Eq.(3.3):

$$(c^2 - x'^2)(1 - \alpha/\tau)^{-4}(x - tx')^{-2} = k^2 = \text{const} \tag{3.4}$$

Separation of variables was now effected in (3.4) by substituting $\tau, \tau_1 = \tau t^{-1}$. It was thus possible to determine a parametric representation of the coordinate and time:

$$x = c\,(2k_1)^{-1}\,(k_1^2 e^{\psi(\tau)} - e^{-\psi(\tau)}), \quad t = (2k_1)^{-1}\,\tau\,(k_1^2 e^{\psi(\tau)} + e^{-\psi(\tau)})$$

or (after allowing for the dependence of the arbitrary constant k_1 on the initial data x_0, t_0)

$$x = \frac{c\tau}{\tau_0}\left(\frac{x_0}{c}\,\mathrm{ch}\,\psi(\tau) + t_0\,\mathrm{sh}\,\psi(\tau)\right) \tag{3.5}$$

$$t = \frac{\tau}{\tau_0}\left(\frac{x_0}{c}\,\mathrm{sh}\,\psi(\tau) + t_0\,\mathrm{ch}\,\psi(\tau)\right), \quad \tau \geqslant \tau_0 = \sqrt{t_0^2 - (x_0/c)^2}$$

Since the integral $\psi(\tau)$ is convergent as $\tau \to \infty$, the functions $\mathrm{sh}\,\psi(\tau)$, $\mathrm{ch}\,\psi(\tau)$ are bounded everywhere. The second of formulae (3.5) shows that as τ varies over the ray $[\tau_0, \infty[$, the variable t also takes all values from t_0 to ∞. This completeness property of the parametric representation (3.5) of the solution is also valid in the full-dimensional case.

In the second step of the solution the one-dimensional motion (3.5) was converted into a two-dimensional one, by adding one more equation to (3.5), namely, $y = 0$, and applying a composition of two transformations to the resulting set of equations: a Lorentz transformation applied to the variables y, t (but not to x), and a rotation of the (x, y) plane.

Introduction of the unit vectors **i**, **j** of the coordinate axes x, y and the vectors $\mathbf{r}_0 = \mathbf{i}x_0 + \mathbf{j}y_0$, $\mathbf{v} = \mathbf{i}x_0' + \mathbf{j}y_0'$ now produced the solution (3.2). When the integral (2.4) exists, this solution is the most general one (since all motions of the point are two-dimensional).

The aforementioned asymptotic properties of the geodesics may be verified directly for the exact solution (3.2):

$$\lim_{t\to\infty} \mathbf{V} = \frac{\lim_{\tau\to\infty}\mathbf{r}_\tau'}{\lim_{\tau\to\infty} t_\tau'} = \frac{\mathbf{r}_0\,\mathrm{ch}\,\psi(\infty) + \boldsymbol{\lambda}_0\,\mathrm{sh}\,\psi(\infty)}{t_0\,\mathrm{ch}\,\psi(\infty) + p_0\,\mathrm{sh}\,\psi(\infty)} = \mathbf{V}_\infty$$

$$\lim_{t\to\infty}(\mathbf{r} - \mathbf{V}t) = \frac{p_0\mathbf{r}_0 - t_0\boldsymbol{\lambda}_0}{k\tau_0\,(t_0\,\mathrm{ch}\,\psi(\infty) + p_0\,\mathrm{sh}\,\psi(\infty))} \equiv \mathbf{a} = \mathbf{const}$$

The non-relativistic approximation to the geodesics may be obtained either directly from the exact solution (3.2), or by direct integration of the equations of the geodesics in the limit of $c \to \infty$. The result is an explicit function of time:

$$\mathbf{r} = \mathbf{b}t + \mathbf{g}t/(t - \alpha), \quad \mathbf{b} = \mathbf{const}, \quad \mathbf{g} = \mathbf{const}$$

The geodesics turn out to be hyperbolae. Their equations in the plane $z = 0$ are:

$$y_1 = \Delta\,(1 - \alpha\Delta/(x_1 - \alpha\Delta)), \quad \Delta = g_2 b_1 - g_1 b_2$$

$$x_1 = g_2 x + g_1 y, \quad y_1 = b_2 x - b_1 y, \quad (b_1, b_2) \equiv \mathbf{b}, \ (g_1, g_2) \equiv \mathbf{g}$$

4. Motion of a material point in a central field. The equation of motion for the case of a central field F_r may be reduced, by transforming to polar coordinates r, φ in the plane of the motion, to the form

$$\frac{d}{dt}\left[\left(1-\frac{\alpha}{\tau}\right)^2\left(1-\left(\frac{V}{c}\right)^2\right)^{-1/2} mr'\right] = \frac{2m\alpha}{\tau^3}\left(1-\frac{\alpha}{\tau}\right)\sqrt{1-\left(\frac{V}{c}\right)^2}\,r + mr\varphi'^2(1-\alpha/\tau)^2(1-(V/c)^2)^{-1/2} + (1-\alpha/\tau^*)^2 F_r$$

The area integral is

$$(1 - \alpha/\tau)^2(1 - (V/c))^{-1/2}r^2\varphi' = l = \text{const}$$

Hence it follows that the law of motion for a Newtonian field, in the non-relativistic approximation $(c \to \infty)$, is

$$d\,[(1 - \alpha/t)^2 r']/dt = 2\alpha t^{-3}\,(1 - \alpha/t)r + l^2\,(1 - \alpha/t)^{-2}r^{-3} - 0.81\gamma M r^{-2}, \quad r^2\varphi'\,(1 - \alpha/t)^{-2} = l \tag{4.1}$$

For a material point describing a circular orbit of radius r_0 with period of revolution $T = 2\pi/\omega$, we have $l = 0.81 r_0^2\omega$.

Because of the cosmological correction in the equations of motion, the trajectories of the point are open curves and the motion is not steady but slowly evolving.

Let us examine the evolution of an originally circular orbit of a material point in the space-time model under consideration here.

Put $t = t_0 + t_1$, $t_0 = \tau^* = 6 \times 10^9$ years. Time will be measured from a time $t_1 = 0$ corresponding to the modern era.

Introduce a small parameter $\varepsilon = 1/t_0$. Then $\alpha\varepsilon = \alpha/t_0 = 0.1$, $t^{-1} = \varepsilon - \varepsilon^2 t_1 + \ldots$. In the first approximation, $1 - \alpha t^{-1} = 0.9 + 0.1\varepsilon t_1$.

We shall find the radius-vector r as an expansion in powers of ε, confining our attention to the first approximation $r = r_0 + \varepsilon\nu(t_1)$. Substitution of r into the first of Eqs.(4.1) yields the following equation for $\nu(t_1)$:

$$d^2\nu(t_1)/dt_1^2 + \omega^2\nu(t_1) = -0.22 r_0 \omega^2 t_1$$

whose general solution is

$$\nu(t_1) = A\cos(\omega t_1 + \delta) - 0.22 r_0 t_1; \quad \delta, A = \text{const}$$

Thus, to a first approximation,

$$r = r_0 + \varepsilon(A\cos(\omega t_1 + \delta) - 0.22\, r_0 t_1) \tag{4.2}$$

It is evident from this formula that the average variation $\langle\Delta r\rangle$ of the magnitude of the radius-vector amounts to a systematic decrease (the incidence of the point on the central body). The fall of the point in meters per century is given by the formula

$$\langle\Delta r\rangle = -0.37 \times 10^{-5} r_0 \tag{4.3}$$

where r_0 is the original radius of the circular orbit in kilometers. The magnitude of the angular velocity of the radius-vector of the point in the first approximation, calculated from the second formula of (4.1), is

$$\varphi^{\cdot} = \omega + \varepsilon\omega(0.22 t_1 - 2Ar_0^{-1}\cos(\omega t_1 + \delta))$$

By (4.3), the radius-vector of the point has a positive angular acceleration on the average; the magnitude of its additional angular revolution due to this acceleration, in angular seconds per century, is

$$\langle\Delta\varphi\rangle = \varepsilon\omega t_1^2/9 = 24''/T \tag{4.4}$$

T is the period of revolution of the point about the central body, expressed in terrestrial years.

To estimate the orders of magnitude produced by computations with formulae (4.3) and (4.4), they were applied to the planets of the solar system. This gave the following figures for the approach of the moon to the Earth, and of the Earth and Jupiter to the sun, in metres per century: 1.1, 550 and 2.7×10^3, respectively.

The additional angular revolution of Mercury and the Earth about the sun, of the moon about the Earth and of Phobus about Mars, turned out to be 43″, 0.24″, 2.9″ and 300″ per century, respectively.

These figures (computed with the aid of planetary data taken from /3/) are apparently not very close to reality, since they take into account only one purely cosmological effect, which is derived, moreover, from a model whose physical applicability to problems involving central motion of points has not been fully substantiated.

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